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LETTER TO THE EDITOR

Two-state model of self-organized criticality

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Abstract. We study a two-state version of the sandpile model of self-organized criticality. Instead of a critical height of stability as in the sandpile model, we introduce a hard core repulsion among different particles at the same position. In the case of a collision particles hop randomly to the nearest neighbours. Critical exponents obtained by numerical simulation show strong disagreement with the values theoretically predicted for the sandpile model but they are close to the numerical values of other models.

Recently there has been seen a lot of interest in the study of self-organized criticality (SOC) introduced by Bak, Tang and Wiesenfeld (BTW) [1]. In this new phenomenon a system under time evolution of its own dynamics reaches a critical state which lacks any characteristic length or time scales and obeys power law distributions. The critical state is independent of the arbitrary initial configuration to start with and unlike ordinary critical phenomena no fine tuning is necessary to arrive at this state. This state is called the self-organized critical state and BTW suggested that this phenomenon may be the underlying cause of a large class of phenomena involving dissipative nonlinear transport in open systems, such as $1/f$ noise in electrical networks, light pulses from quasers etc [1]. However it was demonstrated later that generally the scaling of the lifetimes and sizes does not necessarily lead to a non-trivial power law for the power spectrum other than the $1/f^2$ type variations [2, 3]. Until now experimental works related to SOC have been reported on sandpiles [4], water drops on window panes [5] and magnetic domain structures [6]. However, it should be mentioned that the criterion of the absence of a tuning parameter is also present in many other fractal growth phenomena established in the literature [7, 8]. Recently it has been observed that the local conservation of particle number is not a necessary criterion to achieve the power law distributions [9].

BTW proposed a simple automaton model, popularly known as sandpile model [1], to explain this new phenomenon which is described below. Each site of a d-dimensional lattice is associated with an integer variable h which can be zero or positive. This quantity represents the number of sand grains in a column at that site. Starting from an arbitrary initial distribution of sand heights, one adds sand grains one at a time to a randomly chosen site of the lattice

$$h_i \rightarrow h_i + 1. \qquad (1)$$

There is a critical value of the sand height h_c at all sites and if $h \geq h_c$ at any site that sand column topples. As a result the height at that site reduces as

$$h_i \rightarrow h_i - z. \qquad (2)$$

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This amount of sand is distributed equally within the z nearest neighbours which gain one unit each

$$h_j \to h_j + 1. \tag{3}$$

In this way the particle number is conserved for topplings within the lattice; however, for toppling on the boundary sites some grains fall outside the boundary and never come back. After a long time this inflow (sand addition) and outflow (through the boundary) are equal on the average and the system reaches a steady state which is characterized by a fixed value of the average sand height. For toppling at a site the nearest neighbours gain sand grains. Some of them may reach the critical height h_c which will topple on the next time step. In this way a cascade of topplings occurs forming an avalanche which has no characteristic size s or lifetime t but their distributions obey power law decays [1, 7, 10, 11]

$$D(s) \sim s^{-\tau} \tag{4}$$

$$D(t) \sim t^{-y} \tag{5}$$

and the average lifetime T_s for cluster size s is scaled as

$$T_s \sim s^x. \tag{6}$$

The relation $D(t)\,\mathrm{d}t \sim D(s)\,\mathrm{d}s$ with equations (4)-(6) leads to the scaling relation [2]

$$x(1-y) = 1 - \tau. \tag{7}$$

Here we study a two-state version of the sandpile model. Consider a square lattice where the sites can be either empty or occupied with particles. No more than one particle is allowed to be at a site in the stationary state. One particle is added to one of the randomly chosen sites. If it is empty, it gets occupied by that particle and a new particle is launched. If there is already a particle at that site a 'hard core interaction' throws all the particles out from that site and the particles are redistributed in a random manner among its neighbours. It can happen that some of the neighbours were already occupied; then the particles are again redistributed and so on. In this way cascades are created. A cascade is stopped if no occupancy higher than one is present. Free boundaries are used, i.e. particles can leave the system on the boundaries. For this simulation we follow the cluster growth algorithm for the sandpile model described in [7]. Here h is the occupation number at a given site. We update the system in parallel through the following steps which all together constitute a unit time step:

(*a*) at any instant all collision sites are located;

(*b*) all these sites are made empty;

(*c*) for each particle in each collision site one neighbouring site is randomly selected and the particle number at that site is increased by one;

(*d*) collision sites for the next time step are located from these new sites.

The cluster size s is measured by the total number of collisions in a cascade. One site may have many collisions in different times within the same cascade. The lifetime of a cascade is measured by the number of sweeps needed for the cascade to become quiet. If one starts with an empty lattice the particle density grows and reaches a stationary value p_c. However one can start with a fully occupied lattice and just throw a particle to any site. A large cascade will result and the system will become steady when particle density has come down to p_c.

Compared with the sandpile model we see that in our model $h_c = 2$. The collisions are analogous to the topplings. At a collision particles are distributed randomly to the neighbouring sites in comparison with which each neighbour gets one particle for a toppling-in sandpile model. Therefore the time evolution of the cascade is random compared with the deterministic evolution of sandpile avalanche.

We have studied this model numerically. We always start from an empty lattice and go on throwing particles one after another. Whenever there is a collision, we choose with equal probability a neighbouring site for each particle and the particle is transferred in that location. Initially the particle density grows with time and there is hardly any particle which goes out of the system. However, once the threshold density p_c is reached it no longer changes the average. On the average the net incoming flow rate becomes equal to the outflow. In this situation single particle addition causes cascades of all length scales and both in size and lifetimes.

We first measure the variation of the threshold density with respect to the lattice size. Once we reach the critical state we measure the average density at fixed time intervals over many configurations. Similar to the sandpile model we get a linear fit to all points on a plot with $1/L$ (see figure 1) with a slight curvature at small L values. This $1/L$ behaviour is due to the boundary effect, the average density on the boundary is less than the inner core region of the lattice [7]. After extrapolation to the $L \to \infty$ limit we get a value of $p_c = 0.6832 \pm 0.0010$ for the infinite system.

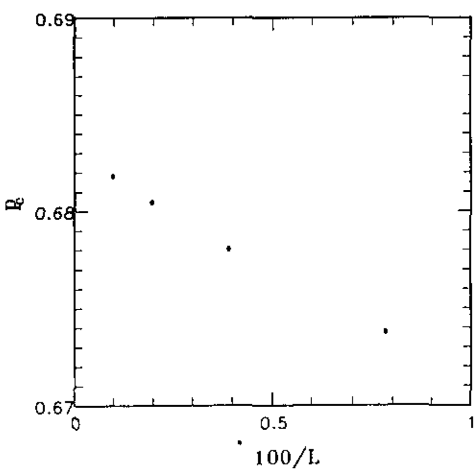

Figure 1. Average threshold density p_c is plotted with respect to $1/L$. The extrapolated value for $L \to \infty$ is 0.6832 ± 0.0010.

At the critical state large numbers of cascades are considered, e.g., one million up to the lattice size $L = 128$ and 190 000, 224 000 and 185 000 for the lattice sizes $L = 256$, 512 and 1024 respectively. We measure the size distribution of these cascades by measuring the number of clusters corresponding to a particular size. These numbers are integrated over bins of lengths increasing exponentially; successive bin lengths vary by a factor $\sqrt{2}$, e.g. 1, 2, 4, 5, 8, 10, 16, 22, 32, etc. For the initial bins successive ratios of bin lengths are far away from $\sqrt{2}$. As a result we get oscillations for small cluster sizes for both cluster size and lifetime distributions. The integrated probability distribution function $D(s)$ obtained in this way is plotted in a double logarithmic scale with size s (see figure 2) for different lattice sizes. We see that for each curve the middle region fits to a straight line but not the very small and very large cascade sizes. These straight portions for different curves fall on the same line. The largest straight portion is obviously for the largest lattice size $L = 1024$. We calculate the cluster size distribution exponent τ by measuring the slope of the straight portion of the curve for $L = 1024$ (which gives us a value of $\tau - 1$) and obtain $\tau = 1.28 \pm 0.02$.

In figure 3 we see a finite-size scaling of the cluster size distribution following Kadanoff *et al* [12]. The cluster size probability distribution $D(s, L)$ for a lattice size L is assumed to follow the scaling law

$$D(s, L) = L^{-\beta_s} f(s/L^{\nu_s}). \tag{8}$$

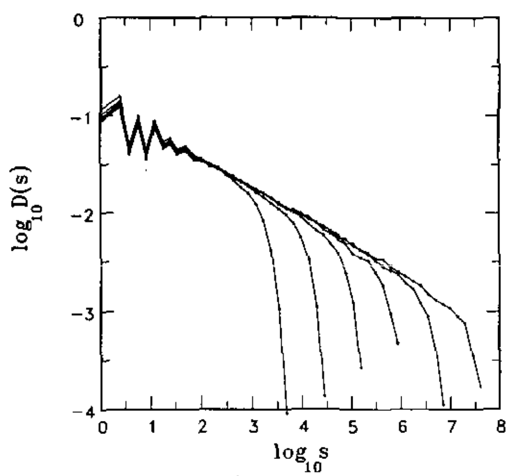

Figure 2. Double logarithmic plot of the cluster size probability distribution $D(s)$ (integrated over bins) against cluster size s. Curves starting from left to right correspond to $L = 32, 64, 128, 256, 512$ and 1024.

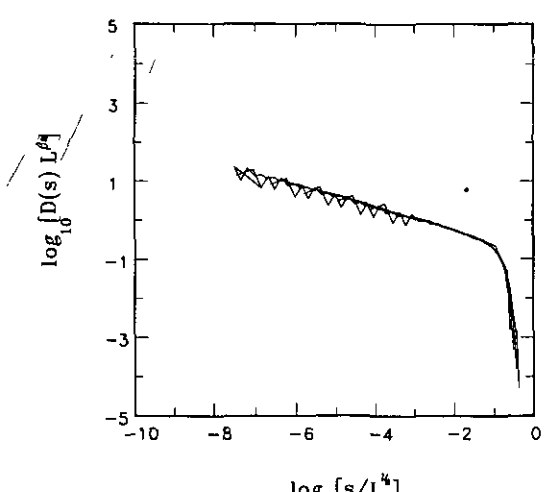

Figure 3. Finite-size scaling plot of integrated cluster size probability distribution $D(s)L^{\beta_s}$ against s/L^{ν_s} in a double logarithmic scale. For the collapse of data shown in this figure we used $\nu_s = 2.75$ and $\beta_s = 0.82$.

We used our data integrated over bins for the cluster distribution $D(s, L)$. We therefore expect that the scaling function $f(x)$ should behave $f(x) \sim x^{-\tau+1}$ for $0 \ll x \ll 1$. As in this range of x values there should not be any strong lattice size dependence (other than a weak dependence of the amplitude on L) we relate the cluster size exponent τ with the scaling exponents as $\tau = (\beta_s + \nu_s)/\nu_s$. We plot $D(s, l)L^{\beta_s}$ against s/L^{ν_s} using a double logarithmic scale in figure 3. We see a very good collapse of the data for all lattice sizes starting from $L = 32$ to 1024 on the same plot. Our best estimate for the exponents are $\nu_s = 2.75$ and $\beta_s = 0.82$. We calculate the value of the exponent τ using these values of β_s and ν_s as 1.30. This value of τ is quite consistent with that obtained before which shows that the scaling works very well.

In a similar way we study the lifetime distribution of the cascades. For big cascades the lifetimes are much smaller than the cluster size because at any time during the cascades a large number of sites have collisions all of which corresponds to a single time step. As a consequence our lifetime distribution is much shorter than the size distributions. On a double logarithmic plot of $D(t)$ against t we still see considerable curvature (see figure 4). Our best estimate for the exponent y is 1.47 ± 0.10.

We also tried a scaling analysis for the integrated lifetime distribution as

$$D(t, L) = L^{-\beta_t} f(t/L^{\nu_t}). \tag{9}$$

Plotting $D(t, L)L^{\beta_t}$ against t/L^{ν_t} on a double logarithmic scale as in figure 5 we see that best data collapse corresponds to $\nu_t = 1.55$ and $\beta_t = 0.78$. These values give the value of $y = 1.50$ which is again consistent with the previous estimate of y.

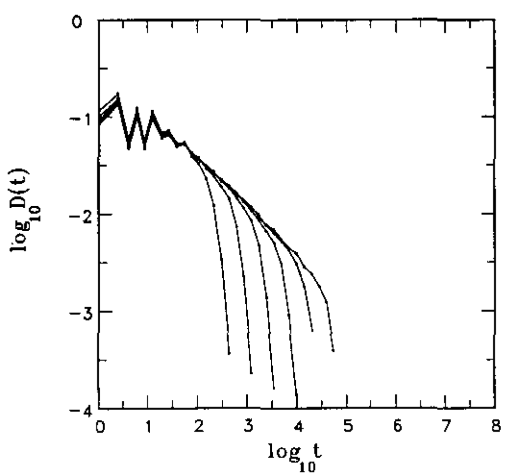

Figure 4. Double logarithmic plot of the cluster lifetime distribution $D(t)$ (integrated over bins) against lifetime t. Curves starting from left to right correspond to $L = 32$, 64, 128, 256, 512 and 1024.

Finally we check whether the exponents τ and y are consistent to each other. For that we calculate the exponent x studying the variation of T_s with s. We calculate the average value of the lifetime T_s for a bin s and plot it in a double logarithmic scale in figure 6. We see that for all lattice sizes the data points fall on one another. The combined curve is a nice straight line. We calculate the value of $x = 0.56 \pm 0.02$. Using the values of $\tau = 1.28$ and $y = 1.47$ and the scaling relation (7) one gets the value of x

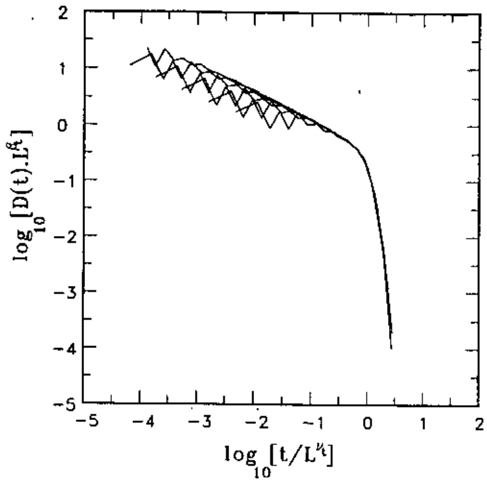

Figure 5. Finite-size scaling plot of integrated cluster lifetime probability distribution $D(t)L^{\beta_t}$ against t/L^{ν_t} in a double logarithmic scale. For the collapse of data shown in this figure we used $\nu_t = 1.55$ and $\beta_t = 0.78$.

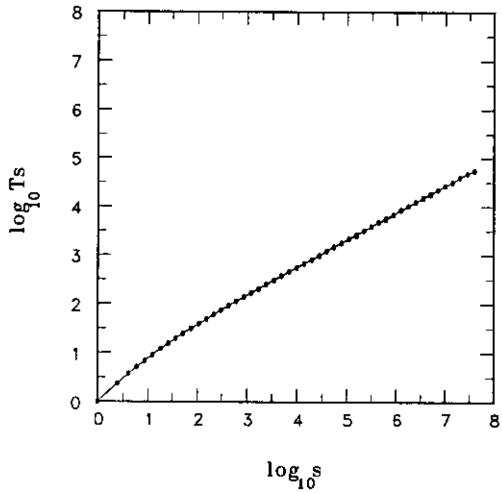

Figure 6. Double logarithmic plot of the average lifetime T_s against cluster size s. The value of the exponent x obtained from the slope is 0.56.

as 0.60. Therefore we see that our estimated exponents are reasonably consistent with one another.

Now we would like to compare our result with the results already known for the sandpile model. Our model is a variation of the sandpile model in which we introduce hard core repulsion between any two particles which forbids two or more particles to occupy the same position simultaneously. Therefore the steady state of the system is when all sites are either occupied or unoccupied. During a collision particles are distributed randomly in difference with the deterministic procedure during a toppling in sandpile model. In spite of these differences we believe that our model should be in the same universality class with the sandpile model. For the sandpile model initial simulation studies [1] and theoretical prediction by Zhang [11] using a continuum version of the model estimates the exponent τ to be 1. Later large-scale simulation of the same model [10] estimated τ to be 1.22. Here we get the value of τ is 1.28 which is far away from the value 1 but close to the value 1.22. The lifetime exponent $y = 1.47$ is to be compared with 1.38 in sandpile model [10]. We believe these differences are results of finite size.

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